Experiment 11: Moments of Inertia

I. About the Experiment

A net force acting on a point mass gives it a linear acceleration, a. For a net force of a particular size, an object with a relatively small mass undergoes a relatively large acceleration, and an object with a large mass undergoes a small acceleration. Mass is the property of an object which tends to resist, or impede, its acceleration. This property is known as (translational) inertia.

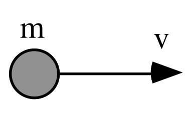
Just as a net force acting on a mass accelerates the mass, a net torque (torque was defined in chapter 4) acting on a rigid body about some axis makes the body accelerate in a rotational manner, or gives it an angular acceleration α. And, just as the mass of an object tends to impede its linear acceleration when acted upon by a net force there is a characteristic property of a rigid body which tends to impede its rotational acceleration about a particular axis when it is acted upon by a net torque. This property is known as rotational inertia and is called the moment of inertia, I.

Although the purpose of this experiment is to measure the moment of inertia of a disk and a ring, the basic physics involved is the principal of conservation of mechanical energy and the relationship of the velocity to the time of a mass moving with constant acceleration. The experimental values of the moments of inertia determined will be compared to calculated theoretical values.

II. Moment of Inertia

Before giving a quantitative description of the moment of inertia, let's apply some simple ideas, with which you should be familiar, in order to have some feeling for what is involved. In many public city parks there are small un-powered merry-go-rounds where you can take small children to play. You do most of the pushing and they do most of the riding. The more kids you have loaded onto the thing, the greater is the work required to get it up to a particular rate of rotation to please the kids. Thus it is evident the rotational inertia depends upon the mass. Now, if you can get all your little friends to move close to the center, you can achieve the same angular velocity with much less effort. Thus the rotational inertia depends not only upon the mass, but how it is distributed in space about the axis of rotation.

To be quantitative, we will use the familiar concept of kinetic energy. A point mass m traveling with a velocity v has a kinetic energy



$$\text{K.E.} = \tfrac{1}{2}mv^2$$

Suppose now that we have a series of point masses attached to an axis with rigid, but massless spokes. For a definite example, suppose we have three masses as shown in Figure 1. The axis of rotation is perpendicular to the x-y plane

At any instant, m_1 is moving with a tangential speed v_1, m_2 is moving with a tangential speed v_2, etc. Thus the total kinetic energy for the system at any instant can be written as

$$\text{K.E.} = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 + \tfrac{1}{2}m_3v_3^2 \qquad \text{Eqn.1}$$

This expression is awkward because each v is different in addition to having each mass different. A simpler expression for the kinetic energy of a rotating body can be obtained by expressing the K.E. in terms of the angular speed ω. If the body is really rigid, all three masses have the same angular speed. That is, they all make the same number of revolutions per second, or more usefully, turn through the same number of radians per second about the axis.

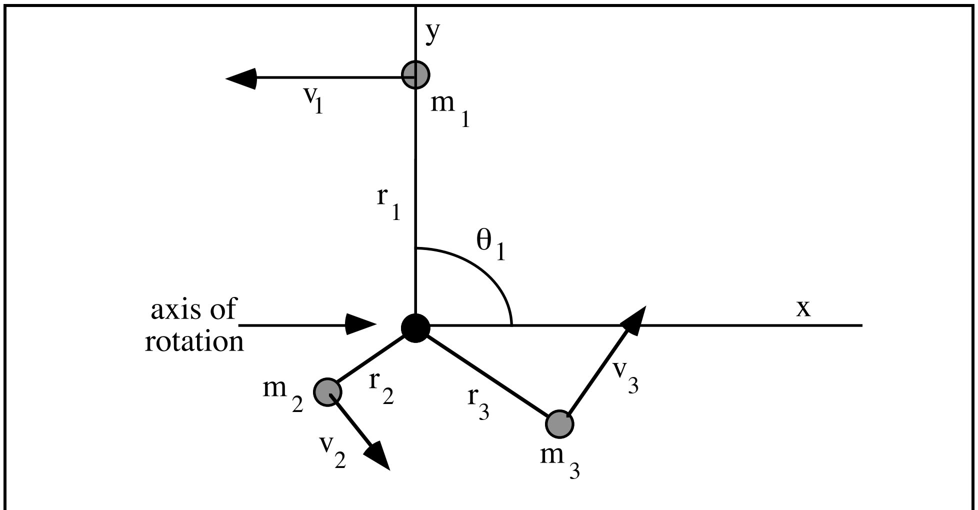


Figure 1

The angular speed ω is defined by

$$\omega = \lim_{\Delta t \to 0}\left(\frac{\Delta\theta}{\Delta t}\right) = \frac{d\theta}{dt} \qquad \text{Eqn.2}$$

If θ is measured in radians then the tangential speed v = ωr (where r is the perpendicular distance from the mass to the **rotational axis**). Thus we can write $v_1 = \omega r_1$, $v_2 = \omega r_2$, and $v_3 = \omega r_3$ (we don't subscript ω's because they are the same). Using these expressions for the tangential speed in Equation 1 we obtain

$$\text{K.E.} = \tfrac{1}{2}m_1\omega^2 r_1^2 + \tfrac{1}{2}m_2\omega^2 r_2^2 + \tfrac{1}{2}m_3\omega^2 r_3^2 \qquad \text{Eqn.3}$$

The factor $\tfrac{1}{2}\omega^2$ occurs in each term, so we can factor it out. Then

$$\text{K.E.} = \tfrac{1}{2}\omega^2\left[m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2\right] \qquad \text{Eqn.4}$$

Now, the bracketed term is what is defined as I, the moment of inertia. Therefore,

$$\text{K.E.} = \tfrac{1}{2}I\omega^2 \qquad \text{Eqn.5}$$

Notice how analogous this is to $K.E. = \frac{1}{2}mv^2$. I and ω play the same role in rotational motion that m and v play in translational motion. (An object which is rolling has both translational and rotational kinetic energy.)

To generalize a bit further, if we have some arbitrary solid body, we can consider it to be made up of a large collection of mass elements, a total of N in all. The first, second and ith mass element m_1, m_2, m_i, and the distance from the elements to the axis r_1, r_2 and r_i are shown in the diagram in Figure 2

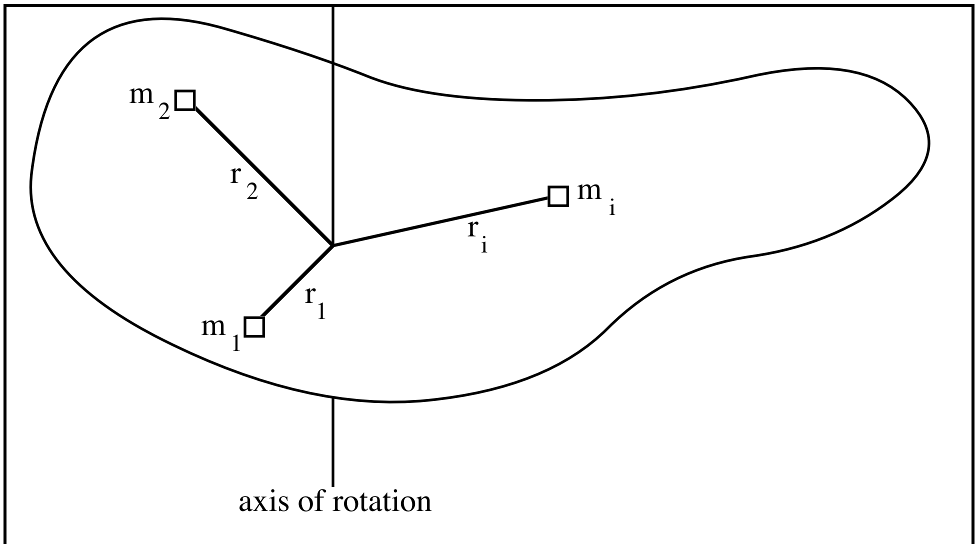


Figure 2

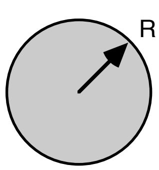
The moment of inertia for such a body is obtained by multiplying each mass element by the square of the distance back to the axis, and taking the sum over all the mass elements

$$I = m_1r_1^2 + m_2r_2^2 + \ldots + m_Nr_N^2 = \sum_{i=1}^{N} m_ir_i^2 \qquad \text{Eqn.6}$$

If the mass is continuously distributed then Equation 6 becomes

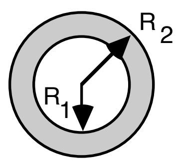
$$I = \int r^2 dm \qquad \text{Eqn.7}$$

The results of making such a computation for a uniform circular disk is:



$$I_{\text{disk}} = \frac{1}{2}MR^2 \qquad \text{Eqn.8}$$

where M is the total mass of the disk and R is its radius. For a uniform cylindrical ring, we have:



$$I_{\mathrm{ring}} = \tfrac{1}{2} M\left(R_1^2 + R_2^2\right) \qquad \text{Eqn.9}$$

In this experiment you will measure the moment of inertia of uniform circular disk and ring. You will then compare your results with the values obtained using Equations 8 and 9.

III. The Experimental Set-up

In this experiment we will make these measurements by using the apparatus shown in Figure 3. With this apparatus we will be able to determine the kinetic energy of a rotating body for a known value of ω. We can then solve Equation 5 for I which gives

$$I = \frac{2(\mathrm{K.E.})}{\omega^2} \qquad \text{Eqn.10}$$

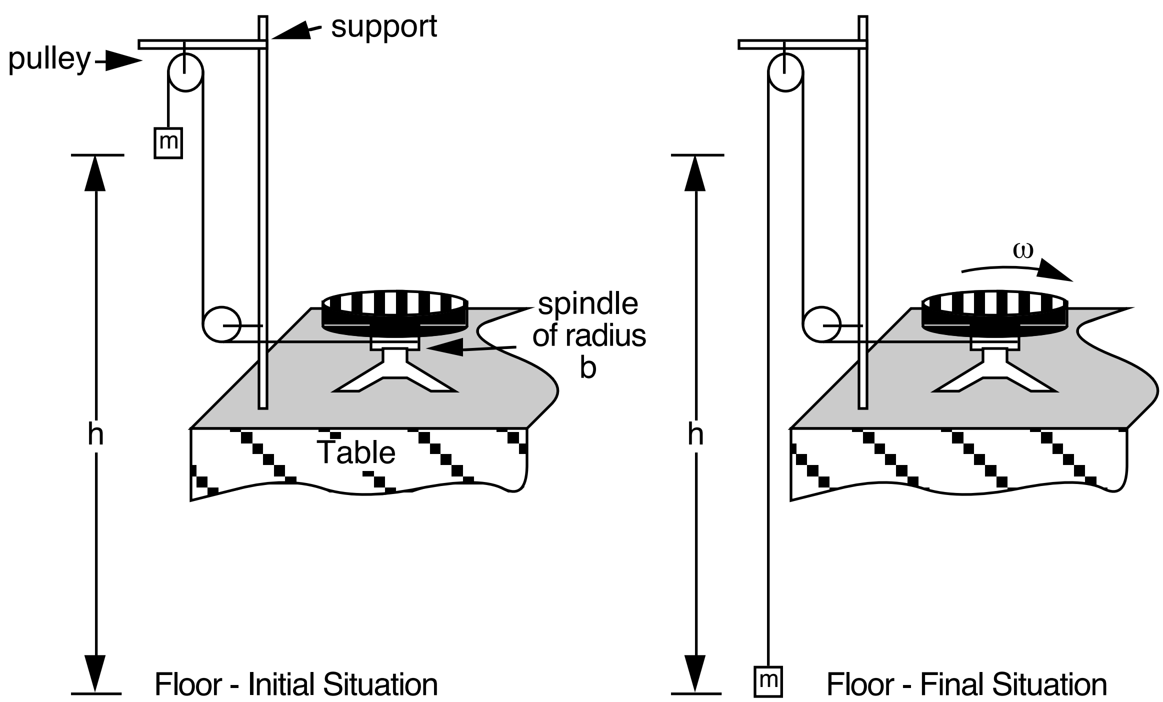


Figure 3

The object whose moment of inertia is to be measured, such as a ring, is placed on top of a supporting structure which is free to rotate. A string is wrapped around the spindle (see Figure 3) of the supporting structure, passed over pulleys, and attached to a mass m which is a height h off the ground. The tension in the string will exert a torque on the spindle causing the entire system, support + ring, to rotate. The system starts from rest. After the mass m is released it descends towards the floor falling through a height h in a time t. The final situation is as shown above with the object in rotational motion, and the mass m in translational motion just before reaching the floor.

We shall now show how we can be determined the moment of inertia of the rotating object from measurements of h, t and m.

To see how the moment of inertia is related to these quantities we make use of the conservation of energy theorem. Suppose for the moment that there is no friction in the system. Then we can write

Initial: $KE_i = 0$ $PE_i = mgh$ $E_i = KE_i + PE_i = mgh$

Final: $KE_f = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$ $PE_f = 0$ $E_f = KE_f + PE_f = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

where E represents the total energy.

Note that we have properly included both the kinetic energy of translation of the mass m and the kinetic energy of rotation for the ring and supporting cross and spindle. If there is no friction in the system then mechanical energy is conserved, $E_i = E_f$, so that

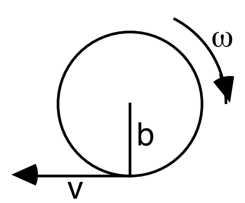
$$mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2 \qquad \text{Eqn.11}$$

This equation is valid if there is no friction in the apparatus so that all of the original potential energy is converted into kinetic energy. But if friction is not negligible, then part of the original potential energy is changed to heat and this part of the original energy cannot be converted to kinetic energy. To put it another way, part of the original potential energy goes to overcome the friction forces in the system. We can find approximately what amount of the initial potential energy of the mass m is lost to friction in this following manner.

Set up the apparatus and, starting with m = 0, add mass until m begins to move and descends with constant velocity. Call the mass needed to overcome friction m_0. Then an amount m_0gh of the initial potential energy is lost to heat so that the amount of potential energy converted into kinetic energy when a mass m descends is $(m-m_0)gh$, not mgh. Thus the energy equation (11) becomes

$$(m - m_0)gh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2 \qquad \text{Eqn.12}$$

In order to use the previous equation to obtain the moment of inertia, I, we need to know the values of v and ω after the mass m has descended a distance h. The speed of the string (and hence of the mass m) is the same as the tangential speed of the spindle. Thus v and ω are related by v = ωb where b is the radius of the spindle.



so

$$\omega^2 = \frac{v^2}{b^2} \qquad \text{Eqn.13}$$

It is not particularly easy to directly measure the final speed v. However, we can obtain v from knowing the time it takes the mass m to fall the distance h and it is easy to measure the time. We obtain v from h and t in the following manner. Since the mass descends with a constant acceleration we have

$$v = v_0 + at \qquad \text{Eqn.14a}$$
$$v^2 = v_0^2 + 2ah \qquad \text{Eqn.14b}$$

We can solve Equation 14a for a and substitute the expression for a into Equation 14b. Since $v_0 = 0$ we obtain Equation 14 for the final speed v.

$$v = \frac{2h}{t} \qquad \text{Eqn.15}$$

We can now replace v and ω in Equation 12 using Equations 13 and 15. We then solve the resulting equation for the moment of Inertia I. This yields

$$I = mb^2\left[\left(\frac{m - m_0}{m}\right)\frac{gt^2}{2h} - 1\right] \qquad \text{Eqn. 16}$$

Equation 16 is simply Equation 10 where both the K.E. and ω have been expressed in terms of measurable parameters. Note that Equation 16 does **not** contain the mass of the rotating object.

The only additional difficulty is that the moment of inertia we obtain this way is $I_{(object\ +\ support)}$. That is it is the sum of the moment of inertia of the support structure ($I_{(support)}$) and the moment of inertia of the object ($I_{(object)}$) that has been placed on the structure. So, we'll also need to determine experimentally the moment of inertia of the support structure. We will then obtain the moment of inertia of the rotating body by subtraction, i.e.

$$I_{(object)} = I_{(object\ +\ support)} - I_{(support)} \qquad \text{Eqn.17}$$

IV. Procedure

Notes:

i) **Record your data in the data table provide on page 8.**

ii) **You need to be particularly careful in your measurements of h and t.**

1. Make sure the axle is vertical by using a level and adjusting the leveling screws. Make sure the string runs smoothly over the pulleys.

2. Using vernier calipers, carefully measure (d) the diameter of the spindle the string is wrapped around (see Figure 3). **Note: b = d/2**.

3. Set up the rotating apparatus so that the driving mass m can fall a distance h ~ 1.5 meters without letting the weight hanger strike the floor.

4. Determine the moment of inertia of the supporting structure by doing the following:
   a. Find m_0 by starting with no driving mass and adding mass until m_0 descends with constant velocity (jiggle the structure to give it some initial rotation). Record m_0.
   b. Use a mass m such that as the mass m accelerate downward it takes about 10 seconds to fall a distance h ~ 1.5 meters
   c. For three trials record the time (t) and distance (h) it takes for m to fall the distance h.

5. Measure the moment of inertia of the combination of the ring and the supporting structure.
   a. Find m_0 by starting with no driving mass and adding mass until m_0 descends with constant velocity (jiggle the structure to give it some initial rotation). Record m_0.
   b. Use a mass m such that as the mass m accelerate downward it takes about 10 seconds to fall a distance h ~ 1.5 meters
   c. For three trials record the time (t) and distance (h) it takes for m to fall the distance h.

6. Repeat 5, except use the disk.

7. Using the large pan balances carefully measure the mass (M) of ring and disk.

8. Determine the **radii** of the ring and disk by measuring their **diameters**.

Data Table

Part 2:	Spindle diameter d = ____________					
Part 4:	I for support structure		m =	_______	m_0 =	_______
	t =	1) _______	2)	_______	3)	_______
	h =	_______				
Part 5:	I for ring and support structure		m =	_______	m_0 =	_______
	t =	1) _______	2)	_______	3)	_______
	h =	_______				
Part 6:	I for disk and support structure		m =	_______	m_0 =	_______
	t =	1) _______	2)	_______	3)	_______
	h =	_______				
Part 7:	M_{Ring} _________		M_{disk} __________			
Part 8:	Ring D_1 = __________		Ring D_2 = __________		Disk D = __________	

V. Calculations and Analysis

Notes: i) Be sure to use units of kilograms and meters in all calculations.

ii) If you set up your spreadsheet to do 1a carefully, you can extend the this spreadsheet for the other two trials by copying the appropriate cells and pasting them further down the spreadsheet.

iii) <u>Great care needs to be taken when entering Equation 16 into the spreadsheet.</u> The best way to do this is to start entering the equation from the inner most part and working out! Large percentage differences (>10%) between the theoretical and measured values are usually due to errors in entering Equation 16.

1. Enter **all** of your data into an Excel spreadsheet and **use** the spreadsheet to calculate:

   a. using the data from part 4 and Equation 16 the experimental value of the moment of inertia of the support structure. For t use the average of the three trials.

   b. using the data from part 5, Equation 16, and the equation $I_{(ring)} = I_{(ring\ +\ support)} - I_{(support)}$ the experimental value of the moment of inertia of the ring. For t use the average of the three trials.

   c. using the data from part 6, Equation 16, and the equation $I_{(disk)} = I_{(disk\ +\ support)} - I_{(support)}$ the experimental value of the moment of inertia of the disk. For t use the average of the three trials.

   d. using Equation 9 the theoretical value of the moment of inertia of the ring.

   e. using Equation 8 the theoretical value of the moment of inertia of the disk.

   f. the percent difference between the theoretical and experimental values.

2. To what do you attribute the differences calculated in part 1f?

V. Questions

1. A hollow cylinder and a solid cylinder having the same mass and diameter are released from rest simultaneously at the top of an inclined plane. Which reaches the bottom first? Explain.

2. Animals that depend on being able to run fast have slender lower legs with flesh and muscle concentrated higher up near the shoulder. Why is this distribution of mass advantageous?

3. Why is it that a child playing tightrope on a railroad track rail finds it easier to maintain her balance when her arms are fully outstretched to the sides?

4. A diatomic molecule can be thought of as something like a weight lifting barbell with most of the mass of the molecule concentrated in two spherical (radius r) mass separated by a distance d>>r.. About what axis through the center of mass of the molecule would the moment of inertia be the least? About what axis through the center of mass of the molecule would the moment of inertia be the greatest?